

Properties of the Mittag-Leffler relaxation function

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The Mittag-Leffler relaxation function, $E_\alpha(-x)$, with $0 \leq \alpha \leq 1$, which arises in the description of complex relaxation processes, is studied. A relation that gives the relaxation function in terms of two Mittag-Leffler functions with positive arguments is obtained, and from it a new form of the inverse Laplace transform of $E_\alpha(-x)$ is derived and used to obtain a new integral representation of this function, its asymptotic behaviour and a new recurrence relation. It is also shown that the fastest initial decay of $E_\alpha(-x)$ occurs for $\alpha = 1/2$, a result that displays the peculiar nature of the interpolation made by the Mittag-Leffler relaxation function between a pure exponential and a hyperbolic function.

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1. Introduction

The Mittag-Leffler function $E_\alpha(z)$, named after its originator, the Swedish mathematician Gösta Mittag-Leffler (1846–1927), is defined by [1–4]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (1)$$

where z is a complex variable and $\alpha \geq 0$ (for $\alpha = 0$ the radius of convergence of equation (1) is finite, and one has by definition $E_0(z) = 1/(1-z)$). The Mittag-Leffler function is a generalization of the exponential function, to which it reduces for $\alpha = 1$, $E_1(z) = \exp(z)$. For $0 < \alpha < 1$ it interpolates between a pure exponential and a hyperbolic function, $E_0(z) = 1/(1-z)$. The precise nature of this interpolation is the subject of the present study. The Mittag-Leffler function obeys the following relations [3]:

$$E_{1/n}(z^{1/n}) = e^z \left[n - \sum_{k=1}^{n-1} \frac{\Gamma(1 - k/n, z)}{\Gamma(1 - k/n)} \right] \quad n = 2, 3, \dots, \quad (2)$$

$$E_{m\alpha}(z^m) = \frac{1}{m} \sum_{k=0}^{m-1} E_{\alpha}(z e^{2\pi i k/m}) \quad m = 2, 3, \dots \quad (3)$$

It follows that $E_{1/2}(z) = \exp(z^2)\text{erfc}(-z)$ and $E_2(z) = \cosh(\sqrt{z})$. An explicit expression for any rational value of the parameter $\alpha = m/n$ can be obtained from equations (2) and (3).

The generalized Mittag-Leffler function is [3]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (4)$$

so that $E_{\alpha,1}(z) = E_{\alpha}(z)$. In the simplest form $\alpha, \beta \geq 0$. Algorithms for the computation of the generalized Mittag-Leffler function were recently discussed [5].

There has been much recent interest in the Mittag-Leffler and related functions in connection with the description of relaxation phenomena in complex physical and biophysical systems [6–18], namely within the framework of fractional (non-integer) kinetic equations. In this work, and having in view the applications of this function to relaxation phenomena, the discussion will be generally restricted to $E_{\alpha}(-x)$ that corresponds to a relaxation function when x is a non-negative real variable (usually standing for the time) and $0 \leq \alpha \leq 1$.

2. Basic relation

Using equations (1) and (4), $E_{\alpha}(-x)$ can be written in terms of two Mittag-Leffler functions with positive arguments,

$$E_{\alpha}(-x) = E_{2\alpha}(x^2) - x E_{2\alpha,1+\alpha}(x^2). \quad (5)$$

A particular case of equation (3) follows immediately from equation (5),

$$E_{2\alpha}(x^2) = \frac{E_{\alpha}(x) + E_{\alpha}(-x)}{2}. \quad (6)$$

It also follows from equation (5) that

$$E_{\alpha}(-i\omega) = E_{2\alpha}(-\omega^2) - i\omega E_{2\alpha,1+\alpha}(-\omega^2), \quad (7)$$

a result that will be used in the next section.

3. Inverse Laplace transform

A simple form of the inverse Laplace transform of $E_\alpha(-x)$ can be obtained by the method outlined in [17]. Briefly, the three following equations can be used for the direct inversion of a function $I(x)$ to obtain its inverse $H(k)$,

$$H(k) = \frac{e^{ck}}{\pi} \int_0^\infty [\text{Re}[I(c + i\omega)] \cos(k\omega) - \text{Im}[I(c + i\omega)] \sin(k\omega)] d\omega, \tag{8}$$

$$H(k) = \frac{2e^{ck}}{\pi} \int_0^\infty \text{Re}[I(c + i\omega)] \cos(k\omega) d\omega \quad k > 0, \tag{9}$$

$$H(k) = -\frac{2e^{ck}}{\pi} \int_0^\infty \text{Im}[I(c + i\omega)] \sin(k\omega) d\omega \quad k > 0. \tag{10}$$

where c is a real number larger than c_0 , c_0 being such that $I(z)$ has some form of singularity on the line $\text{Re}(z) = c_0$ but is analytic in the complex plane to the right of that line, i.e., for $\text{Re}(z) > c_0$.

Using equation (7), application of equation (9) to $E_\alpha(-x)$ with $c = 0$, implying $0 \leq \alpha \leq 1$, yields a general relation for its inverse Laplace transform $H_\alpha(k)$,

$$H_\alpha(k) = \frac{2}{\pi} \int_0^\infty E_{2\alpha}(-\omega^2) \cos(k\omega) d\omega \quad k > 0, \quad 0 \leq \alpha \leq 1, \tag{11}$$

hence, for instance

$$H_1(k) = \frac{2}{\pi} \int_0^\infty \cosh(i\omega) \cos(k\omega) d\omega = \frac{2}{\pi} \int_0^\infty \cos(\omega) \cos(k\omega) d\omega = \delta(k - 1), \tag{12}$$

$$H_{1/2}(k) = \frac{2}{\pi} \int_0^\infty e^{-\omega^2} \cos(k\omega) d\omega = \frac{1}{\sqrt{\pi}} e^{-k^2/4}, \tag{13}$$

$$H_{1/4}(k) = \frac{2}{\pi} \int_0^\infty e^{\omega^4} \text{erfc}(\omega^2) \cos(k\omega) d\omega, \tag{14}$$

$$H_0(k) = \frac{2}{\pi} \int_0^\infty \frac{\cos(k\omega)}{1 + \omega^2} d\omega = e^{-k}. \tag{15}$$

Another integral representation of $H_\alpha(k)$, based on the Lévy one-sided distribution $L_\alpha(k)$ [8], was obtained by Pollard [19] (see also [17,18]),

$$H_\alpha(k) = \frac{1}{\alpha} k^{-(1+\frac{1}{\alpha})} L_\alpha\left(k^{-\frac{1}{\alpha}}\right). \tag{16}$$

If equation (8) is used instead of equation (9), and taking into account equation (7),

$$H_\alpha(k) = \frac{1}{\pi} \int_0^\infty [E_{2\alpha}(-\omega^2) \cos(k\omega) + \omega E_{2\alpha,1+\alpha}(-\omega^2) \sin(k\omega)] d\omega \quad 0 \leq \alpha \leq 1. \quad (17)$$

This form is less simple than equation (11), but is (formally) necessary for finding the asymptotic expansion of $E_\alpha(-x)$, as will be done in Section 5.

4. Complete monotonicity

It is known [19,20] that $E_\alpha(-x)$ is completely monotonic for $x \geq 0$ if $0 \leq \alpha \leq 1$, i.e., that

$$(-1)^n \frac{d^n E_\alpha(-x)}{dx^n} \geq 0, \quad x \geq 0, \quad 0 \leq \alpha \leq 1. \quad (18)$$

We remark here that this result follows immediately from

$$(-1)^n \frac{d^n E_\alpha(-x)}{dx^n} = \int_0^\infty k^n H_\alpha(k) e^{-kx} dk, \quad (19)$$

by noting that $H_\alpha(k) > 0$ for $k \geq 0$ if $0 \leq \alpha \leq 1$ ($H_\alpha(k)$ is a probability density function), as could be conjectured simply by plotting the function $H_\alpha(k)$ for several values of α . Demonstration that $H_\alpha(k) > 0$ for $k \geq 0$ is direct for $0 \leq \alpha \leq 1/2$, using equation (11) and knowing that $E_\alpha(-x)$ ($\alpha \leq 1$) is always positive and decreases monotonically. General demonstrations for $0 \leq \alpha \leq 1$ were given by Feller [20] and Pollard [19].

5. Behaviour near the origin

Any relaxation function $I(x)$ can be written as

$$I(x) = \exp\left(-\int_0^x k(u) du\right), \quad (20)$$

where $k(x)$ is a x -dependent rate coefficient. When the relaxation is exponential, $k(x)$ is obviously constant. For the Mittag-Leffler relaxation function,

$$k(x) = -\frac{d}{dx} \ln E_\alpha(-x) = \frac{1}{E_\alpha(-x)} \sum_{n=0}^{\infty} \frac{(n+1)(-x)^n}{\Gamma(1+\alpha+\alpha n)}, \quad (21)$$

whose initial value is finite and close to unity,

$$k(0) = \int_0^\infty k H_\alpha(k) dk = \frac{1}{\Gamma(1+\alpha)}. \quad (22)$$

It follows nevertheless from equation (22) that the fastest initial decay occurs for $\alpha = 1/2$, a result hitherto unnoticed, and that shows the peculiar nature of the interpolation between a pure exponential and a hyperbolic function performed by the Mittag-Leffler relaxation function.

6. Asymptotic behaviour

Expansion of equation (17) in a power series gives

$$H_\alpha(k) = \frac{1}{\pi} \sum_{n=0}^{\infty} a_n(\alpha) k^n, \quad 0 \leq \alpha < 1, \quad (23)$$

with

$$a_0(\alpha) = \int_0^{\infty} E_{2\alpha}(-\omega^2) d\omega. \quad (24)$$

The Laplace transform of equation (23) is the asymptotic expansion of $E_\alpha(-x)$,

$$E_\alpha(-x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{a_n(\alpha)}{x^{n+1}}, \quad 0 \leq \alpha < 1. \quad (25)$$

Since $a_0(\alpha) \neq 0$ for $0 \leq \alpha < 1$, the Mittag-Leffler relaxation function has a hyperbolic (x^{-1}) asymptotic decay for $0 \leq \alpha < 1$, and an exponential decay only for $\alpha = 1$. The cross-over between the initial exponential-like decay and the asymptotic hyperbolic decay occurs at a value of x that is the shorter, the smaller the α . It follows from equation (25) that $E_\alpha(-x^2)$ asymptotically decays as x^{-2} , hence equation (24) is clearly convergent for $0 \leq \alpha < 1/2$.

7. A recurrence relation

By taking the Laplace transform of equation (11), a recurrence relation is obtained,

$$E_\alpha(-x) = \frac{2x}{\pi} \int_0^{\infty} \frac{E_{2\alpha}(-\omega^2)}{x^2 + \omega^2} d\omega, \quad 0 \leq \alpha \leq 1. \quad (26)$$

This relation works in the opposite way of equation (6), and allows the direct calculation of $E_\alpha(-x)$ from $E_{2\alpha}(-x^2)$. In this way, it follows, for instance, that

$$E_1(-x) = \frac{2x}{\pi} \int_0^{\infty} \frac{\cosh(i\omega)}{x^2 + \omega^2} d\omega = e^{-x}, \quad (27)$$

$$E_{1/2}(-x) = \frac{2x}{\pi} \int_0^{\infty} \frac{e^{-\omega^2}}{x^2 + \omega^2} d\omega = e^{x^2} \operatorname{erfc}(x), \quad (28)$$

$$E_{1/4}(-x) = \frac{2x}{\pi} \int_0^\infty \frac{e^{\omega^4} \operatorname{erfc}(-\omega^2)}{x^2 + \omega^2} d\omega. \tag{29}$$

The asymptotic behaviour of $E_\alpha(-x)$ also follows directly from equation (26) for large x ($\alpha < 1$).

8. Integral representations

The starting point is the known Laplace transform of $E_\alpha(-x^\alpha)$, $J_\alpha^\alpha(s)$, which can be obtained in closed form directly from the definition,

$$J_\alpha^\alpha(s) = \int_0^\infty E_\alpha(-x^\alpha) e^{-sx} dx = \frac{s^{\alpha-1}}{1 + s^\alpha}. \tag{30}$$

Application of inversion equation (9) to equation (30) yields

$$E_\alpha(-x^\alpha) = \frac{2}{\pi} \sin(\alpha\pi/2) \int_0^\infty \frac{\omega^{\alpha-1} \cos(x\omega)}{1 + 2\omega^\alpha \cos(\alpha\pi/2) + \omega^{2\alpha}} d\omega, \tag{31}$$

hence a new integral representation for the Mittag-Leffler relaxation function is

$$E_\alpha(-x) = \frac{2}{\pi} \sin(\alpha\pi/2) \int_0^\infty \frac{\omega^{\alpha-1} \cos(x^{1/\alpha}\omega)}{1 + 2\omega^\alpha \cos(\alpha\pi/2) + \omega^{2\alpha}} d\omega. \tag{32}$$

Analogous representations are obtained with equations (8) and (10).

The previously known integral representation for $E_\alpha(-x^\alpha)$,

$$E_\alpha(-x^\alpha) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{k^{\alpha-1}}{1 + 2k^\alpha \cos(\alpha\pi) + k^{2\alpha}} e^{-xk} dk, \tag{33}$$

was obtained from the Bromwich inversion integral [5]. It follows from equation (33) that

$$E_\alpha(-x) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{k^{\alpha-1}}{1 + 2k^\alpha \cos(\alpha\pi) + k^{2\alpha}} e^{-x^{1/\alpha}k} dk. \tag{34}$$

Performing an integration by parts, equation (34) can be rewritten as

$$E_\alpha(-x) = 1 - \frac{1}{2\alpha} + \frac{x^{1/\alpha}}{\pi} \int_0^\infty \arctan\left(\frac{k^\alpha + \cos(\alpha\pi)}{\sin(\alpha\pi)}\right) e^{-x^{1/\alpha}k} dk. \tag{35}$$

Equation (34) can be used to compute the numerical coefficient of the leading term of the asymptotic expansion of $E_\alpha(-x)$. Equation (24) becomes

$$a_0(\alpha) = \frac{\alpha}{\pi} \Gamma(\alpha) \sin(2\alpha\pi) \int_0^\infty \frac{k^{\alpha-1}}{1 + 2k^{2\alpha} \cos(2\alpha\pi) + k^{4\alpha}} dk, \quad \alpha < \frac{1}{2}. \tag{36}$$

9. Conclusions

The Mittag-Leffler relaxation function, $E_\alpha(-x)$, with $0 \leq \alpha \leq 1$, which arises in the description of complex relaxation processes, was studied. From equation (5) that gives the relaxation function in terms of two Mittag-Leffler functions with positive arguments, the inverse Laplace transform of $E_\alpha(-x)$ was obtained, equation (11), and used to derive a new integral representation of this function, equation (32), its asymptotic behaviour, equations (25) and (36), and a new recurrence relation, equation (26). It was also shown that the fastest initial decay of $E_\alpha(-x)$ occurs for $\alpha = 1/2$, a result hitherto unnoticed, and that shows the peculiar nature of the interpolation between a pure exponential and a hyperbolic function performed by the Mittag-Leffler relaxation function.

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